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Integrable three-body problems with two- and three-body interactions

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Abstract. Spin generalizations of both the elliptic Calogero–Marchioro–Wolfes model and the nonlinear Schrödinger model are studied. These models are three-body problems with two- and three-body potentials, and mathematically related with the exceptional root system of type G_2. We construct the integrable differential-difference operator, the so-called Dunkl operator, based on the infinite-dimensional representation for solutions of the variant of the classical Yang–Baxter equation. By use of these operators, we investigate the integrability and the scattering matrices.

1. Introduction

The quantum one-dimensional three-body problem interacting via two- and three-body inverse square potentials was first introduced by Calogero and Marchioro [2], and Wolfes [1]. This model is called the Calogero–Marchioro–Wolfes (CMW) model. They considered the energy spectrum and the scattering matrix for the rational potential. Later Gaudin considered the identical Boson systems interacting via a δ-function potential, and clarified the role of the root systems in integrable many-body problems [3, 4]. From this viewpoint, the integrable three-body problem with two- and three-body interactions is associated with the exceptional Lie algebra of type G_2.

In recent studies, the differential-difference operator has been shown to play a crucial role in the inverse square interaction models [5–7]. We mean the differential-difference operator as a mixture of a differential and a reflection operator. This operator was originally introduced as operators associated with the root systems, and is called the Dunkl operator in recent mathematical terminology [8].

In this paper we propose the spin generalization of both the CMW system with elliptic interaction and the nonlinear Schrödinger (NLS) model with two- and three-body interactions. Their Hamiltonians are respectively defined by

$$\mathcal{H}_{\mathrm{CMW}} = -\sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \wp(\kappa z_{ij})(a^2 - a\kappa P_{ij}) + 3 \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \wp(\tilde{z}_{ij})(b^2 - b\tilde{P}_{ij}) \tag{1.1}$$

$$\mathcal{H}_{\mathrm{NLS}} = -\sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} + a \sum_{\substack{i,j=1\\ i\neq j}}^{3} \delta(z_{ij}) P_{ij} + 3b \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \delta(\tilde{z}_{ij}) \tilde{P}_{ij}. \tag{1.2}$$

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It will be clarified that the elliptic CMW model $\mathcal{H}_{\mathrm{CMW}}$ (1.1) is integrable when $\kappa = 1$ or $\kappa = 3$. Here $\wp(z)$ is the Weierstrass $\wp$-function, and we use conventional notations such as $z_{ij} \equiv z_i - z_j$ and $\tilde{z}_{ij} \equiv z_i + z_j - 2z_k$ for $i \neq j \neq k \neq i$, where in the latter index k is suppressed since it is determined by i and j. Parameters a and b denote coupling constants. Operators P_{jk} and $\tilde{P}_{jk}$ are spin operators and satisfy the following relations [9]:

$$\begin{aligned} &P_{jk} = P_{kj} \qquad P_{jk}P_{kl} = P_{kl}P_{jl} = P_{jl}P_{jk} \qquad P_{jk}^2 = \mathbb{1} \\ &\tilde{P}_{jk} = \tilde{P}_{kj} \qquad \tilde{P}_{jk}\tilde{P}_{kl} = \tilde{P}_{kl}\tilde{P}_{jl} = \tilde{P}_{jl}\tilde{P}_{jk} \qquad \tilde{P}_{jk}^2 = \mathbb{1} \\ &P_{jk}\tilde{P}_{kl} = \tilde{P}_{kl}P_{jl} = \tilde{P}_{jl}P_{jk} \qquad P_{jk}\tilde{P}_{jk} = \tilde{P}_{jk}P_{jk} \qquad \text{for } j,k,l \text{ are distinct.} \end{aligned} \tag{1.3}$$

These operators are the representation of the Weyl group of the exceptional Lie algebra G_2, and in terms of the Pauli spin matrices σ^α ($\alpha = x, y, z$) they may be chosen as

$$P_{jk} = \tfrac{1}{2}(1 + \sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y + \sigma_j^z\sigma_k^z) \qquad \tilde{P}_{jk} = P_{jk}\sigma_1^z\sigma_2^z\sigma_3^z.$$

The trigonometric limit of the spin CMW model $\mathcal{H}_{\mathrm{CMW}}$ (1.1) with $\kappa = 1$ was first studied in [9, 10].

This paper is organized as follows. In section 2, we consider the classical Yang–Baxter equation (cYBE) associated with the root system of type G_2. We regard solutions as operators acting on the functional spaces, and propose two sets of solutions: elliptic and singular. In section 3, following the scheme studied in [11], we shall construct mutually commuting differential-difference operators, called the Dunkl operators, by use of the solutions of the cYBE. Owing to the commutativity of the Dunkl operators, it is straightforward to prove the integrability of both the elliptic CMW model (1.1) and the NLS model (1.2). In section 4, we shall investigate the scattering matrices of the NLS model (1.2) by diagonalizing the Dunkl operator. Section 5 is devoted to the concluding remarks.

2. Classical Yang–Baxter equation

We shall study operator-valued solutions of a variant of the classical Yang–Baxter equation, which is associated with the root system of type G_2. A set of the cYBE is defined as follows [12]:

$$\begin{aligned} &[r^{12}(\xi_{12}), r^{13}(\xi_{13})] + [r^{12}(\xi_{12}), r^{23}(\xi_{23})] + [r^{13}(\xi_{13}), r^{23}(\xi_{23})] \\ &\quad + [r^{12}(\xi_{12}), \tilde{r}^{23}(\tilde{\xi}_{23}) + \tilde{r}^{13}(\tilde{\xi}_{13})] + [r^{23}(\xi_{23}), \tilde{r}^{13}(\tilde{\xi}_{13}) + \tilde{r}^{12}(\tilde{\xi}_{12})] \\ &\quad + [r^{31}(\xi_{31}), \tilde{r}^{12}(\tilde{\xi}_{12}) + \tilde{r}^{23}(\tilde{\xi}_{23})] = 0 \end{aligned} \tag{2.1a}$$

$$[\tilde{r}^{23}(\tilde{\xi}_{23}), \tilde{r}^{13}(\tilde{\xi}_{13})] + [\tilde{r}^{13}(\tilde{\xi}_{13}), \tilde{r}^{12}(\tilde{\xi}_{12})] + [\tilde{r}^{12}(\tilde{\xi}_{12}), \tilde{r}^{23}(\tilde{\xi}_{23})] = 0 \tag{2.1b}$$

$$[r^{jk}(\xi_{jk}), \tilde{r}^{jk}(\tilde{\xi}_{jk})] = 0. \tag{2.1c}$$

Here we call ξ_j the spectral parameters. We have also used the same notations as before, $\xi_{ij} = \xi_i - \xi_j$ and $\tilde{\xi}_{ij} = \xi_i + \xi_j - 2\xi_k$ for $i \neq j \neq k \neq i$. Two sets of operators, $r(\xi)$ and $\tilde{r}(\xi)$, are associated with roots of type G_2 as is drawn in figure 1. While $r^{jk}(\xi)$ acts as $r(\xi)$ on the jth and the kth spaces and as trivial on the other space, $\tilde{r}^{jk}(\xi)$ acts on all spaces, symmetrically on the referred spaces. The second equation (2.1*b*) corresponds to the ordinary cYBE [13], i.e. associated with the root system of type A_n ($n \geqslant 2$). We suppose that $r(\xi)$ and $\tilde{r}(\xi)$ satisfy the unitarity condition and the symmetric condition, respectively:

$$r^{jk}(\xi) = -r^{kj}(-\xi) \tag{2.2}$$

$$\tilde{r}^{jk}(\xi) = \tilde{r}^{kj}(\xi). \tag{2.3}$$

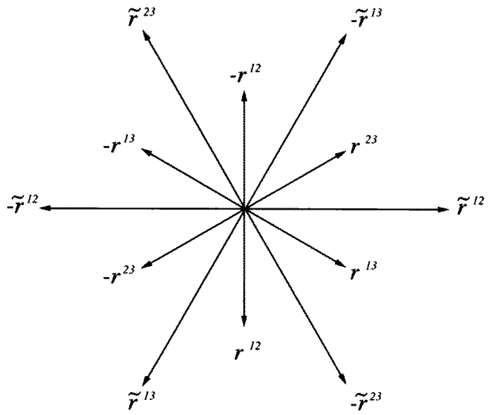

Figure 1. The root system of type G_2 and the classical Yang–Baxter equation.

We shall study the infinite-dimensional representation for solutions of the cYBE (2.1). To this end we regard $r^{jk}(\xi)$ and $\tilde{r}^{jk}(\xi)$ $(j, k = 1, 2, 3)$ as operators acting on the functional space in the following way:

$$r^{jk}(\xi) = F(\xi, z_{jk})K_{jk} \tag{2.4}$$

$$\tilde{r}^{jk}(\xi) = G(\xi, \tilde{z}_{jk})L_{jk} \tag{2.5}$$

where functions $F(\xi, z)$ and $G(\xi, z)$ are to be determined so as to satisfy the cYBE (2.1). Operator K_{jk} denotes an exchange of the jth and the kth coordinates,

$$K_{jk}z_j = z_k K_{jk}. \tag{2.6}$$

Operator L_{jk} acts on the functional space as

$$L_{jk}z_j = (2R - z_k)L_{jk} \qquad L_{jk}z_l = (2R - z_l)L_{jk} \qquad \text{for } j \neq k \neq l \neq j \tag{2.7}$$

where $R \equiv (z_1 + z_2 + z_3)/3$ is the centre-of-mass coordinate. It is noted [10] that the operator L_{jk} can be written as $L_{jk} = K_{jk}I_r = I_r K_{jk}$, where I_r is the inversion operator in relative-coordinate space, $I_r z_j = (2R - z_j)I_r$. The operators K_{jk} and L_{jk} form the dihedral group D_6 and are the same as the Weyl group of the exceptional Lie algebra G_2. One sees that they satisfy the following relations (j, k, l are distinct):

$$\begin{aligned} &K_{jk}K_{kl} = K_{kl}K_{jl} = K_{jl}K_{jk} \qquad K_{jk}^2 = \mathbb{1} \\ &L_{jk}L_{kl} = L_{kl}L_{jl} = L_{jl}L_{jk} \qquad L_{jk}^2 = \mathbb{1} \\ &L_{jk}K_{kl} = L_{kl}K_{jl} = L_{jl}K_{jk} \qquad K_{jk}L_{jk} = L_{jk}K_{jk}. \end{aligned} \tag{2.8}$$

We remark that these operators commute with the spectral parameters $\{\xi_1, \xi_2, \xi_3\}$.

When we substitute the forms of the classical r-operator (2.4) and $\tilde{r}$-operator (2.5) into the cYBE (2.1), we get a set of functional equations,

$$F(-\xi_1, -z_1)F(\xi_2, -z_{12}) + F(\xi_{12}, z_{12})F(-\xi_1, -z_2) + F(\xi_2, z_2)F(\xi_{12}, z_1) = 0 \tag{2.9a}$$

$$G(-\xi_1, -z_1)G(\xi_2, -z_{12}) + G(\xi_{12}, z_{12})G(-\xi_1, -z_2) + G(\xi_2, z_2)G(\xi_{12}, z_1) = 0 \tag{2.9b}$$

$$\begin{aligned} &F(\xi_{12}, z_{12})G(\tilde{\xi}_{13}, \tilde{z}_{23}) + F(\xi_{23}, z_{23})G(\tilde{\xi}_{12}, \tilde{z}_{13}) + F(\xi_{31}, z_{31})G(\tilde{\xi}_{23}, \tilde{z}_{12}) \\ &\quad = F(\xi_{12}, z_{31})G(\tilde{\xi}_{23}, \tilde{z}_{23}) + F(\xi_{23}, z_{12})G(\tilde{\xi}_{13}, \tilde{z}_{13}) + F(\xi_{31}, z_{23})G(\tilde{\xi}_{12}, \tilde{z}_{12}). \end{aligned} \tag{2.9c}$$

The function $F(\xi, z)$ is odd due to the unitarity condition of the classical r-operator (2.2),

$$F(\xi, z) = -F(-\xi, -z). \tag{2.10}$$

We see that both functions $F(\xi, z)$ and $G(\xi, z)$ satisfy the same functional equation, (2.9*a*) and (2.9*b*). This functional equation has appeared in various stages in connection with the integrable systems, and it is well known that solutions are classified as follows [11, 14, 15]:

$$f(\xi, z) = \begin{cases} \alpha\sigma_\xi(z) & \text{elliptic} \\ \alpha(\pi\cot(\pi z) - \pi\cot(\pi\xi)) & \text{trigonometric} \\ \alpha(z^{-1} - \xi^{-1}) & \text{rational} \\ \alpha(\varepsilon(z) - \coth(\xi)) & \text{singular.} \end{cases} \tag{2.11}$$

Here α is an arbitrary constant, and function $\sigma_\xi(z) \equiv \sigma_\xi(z; \tau)$ is an elliptic function defined by

$$\sigma_\xi(z; \tau) = \frac{\vartheta_1(z - \xi; \tau)\vartheta_1'(0; \tau)}{\vartheta_1(z; \tau)\vartheta_1(-\xi; \tau)}$$

where $\vartheta_1(z; \tau)$ is the Jacobi theta function [16],

$$\vartheta_1(z; \tau) = -\sum_{n\in\mathbb{Z}} \exp(\mathrm{i}\pi(n + \tfrac{1}{2})^2\tau + 2\pi\mathrm{i}(n + \tfrac{1}{2})(z + \tfrac{1}{2})) \qquad \Im\tau > 0.$$

The function $\varepsilon(z)$ denotes a signature of z,

$$\varepsilon(z) = \begin{cases} +1 & \text{for } z > 0 \\ -1 & \text{for } z < 0. \end{cases}$$

Since both the trigonometric and the rational solutions are degenerate cases of the elliptic solution, we treat the elliptic and the singular solutions in the following.

We find that, to satisfy the third functional equation (2.9*c*), two functions $F(\xi, z)$ and $G(\xi, z)$ should be fixed as follows:

(i) elliptic solution,

$$\begin{cases} F(\xi, z) = \alpha\sigma_\xi(3z) \\ G(\xi, z) = \beta\sigma_\xi(z) \end{cases} \quad \text{or} \quad \begin{cases} F(\xi, z) = \alpha\sigma_{3\xi}(z) \\ G(\xi, z) = \beta\sigma_\xi(z). \end{cases} \tag{2.12}$$

(ii) singular solution,

$$\begin{cases} F(\xi, z) = \alpha(\varepsilon(z) - \coth(\xi)) \\ G(\xi, z) = \beta(\varepsilon(z) - \coth(\xi)). \end{cases} \tag{2.13}$$

Here α and β are arbitrary parameters. The validity can be checked by use of identities such as

$$\sigma_\lambda(z)\sigma_\mu(w) = \sigma_{\lambda+\mu}(w)\sigma_\lambda(z - w) + \sigma_\mu(w - z)\sigma_{\lambda+\mu}(z) \tag{2.14a}$$

$$\sigma_\mu(z)\sigma_{-\mu}(z) = \wp(z) - \wp(\mu) \tag{2.14b}$$

$$1 + \varepsilon(x)\varepsilon(y) = \varepsilon(x + y)(\varepsilon(x) + \varepsilon(y)). \tag{2.14c}$$

3. Integrable three-body problems

In this section, we shall construct the integrable three-body problems interacting via two- and three-body potentials. For this purpose we shall construct a family of the mutually commuting operators by use of solutions (2.12) and (2.13) of the cYBE (2.1). We introduce three operators $d_j(\xi) \equiv d_j(\xi_1, \xi_2, \xi_3)$ for $j = 1, 2, 3$ as

$$d_j(\xi) = \sum_{\substack{k=1 \\ k\neq j}}^{3} r^{jk}(\xi_{jk}) + \sum_{\substack{k=1 \\ k\neq j}}^{3} \tilde{r}^{jk}(\tilde{\xi}_{jk}) - \sum_{\substack{k,l=1 \\ j\neq k\neq l\neq j}}^{3} \tilde{r}^{kl}(\tilde{\xi}_{kl}). \tag{3.1}$$

Under the conditions (2.2) and (2.3) of the r- and $\tilde{r}$-operators, one sees that they commute each other:

$$[d_j(\xi), d_k(\xi)] = 0. \tag{3.2}$$

Further, we define three differential-difference operators as

$$\hat{d}_j(\xi) \equiv -\mathrm{i}\frac{\partial}{\partial z_j} + d_j(\xi). \tag{3.3}$$

Owing to the commutativity $[\partial/\partial z_j, d_k(\xi)] = 0$, one can conclude that operators $\hat{d}_j(\xi)$ commute each other:

$$[\hat{d}_j(\xi), \hat{d}_k(\xi)] = 0 \qquad \text{for } j, k = 1, 2, 3. \tag{3.4}$$

These integrable differential-difference operators are the Dunkl operators associated with the root system of type G_2.

As the Dunkl operators $\hat{d}_j(\xi)$ constitute an integrable family, we can define mutually commuting integrable operators by

$$\mathcal{I}_n(\xi) = \sum_{j=1}^{3} \pi(\hat{d}_j^n(\xi)). \tag{3.5}$$

The projection π denotes the restriction of the functional space into symmetric space under K_{jk} and L_{jk}:

$$\pi(\mathcal{O}K_{jk}) = \mathcal{O}P_{jk} \qquad \pi(\mathcal{O}L_{jk}) = \mathcal{O}\tilde{P}_{jk}. \tag{3.6}$$

3.1. The CMW model

We use the elliptic solutions (2.12) in the Dunkl operators $\hat{d}_j(\xi)$ (equation(3.3)). In this case the first two conserved operators $\mathcal{I}_n$ are calculated explicitly as follows:

$$\mathcal{I}_1 = -\mathrm{i}\sum_{j=1}^{3} \frac{\partial}{\partial z_j} \tag{3.7}$$

$$\mathcal{I}_2 = -\sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \wp(\kappa z_{ij})(a^2 - a\kappa P_{ij}) + 3\sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \wp(\tilde{z}_{ij})(b^2 - b\tilde{P}_{ij}) \tag{3.8}$$

where we redefined parameters as $\alpha = \mathrm{i}a$ and $\beta = \mathrm{i}b$. Parameter κ is chosen as $\kappa = 1$ or $\kappa = 3$. One sees that operator $\mathcal{I}_2$ coincides with the Hamiltonian of the elliptic CMW model (1.1). This fact proves the quantum integrability of the elliptic CMW model. We note that to eliminate the dependence of spectral parameters in the above calculation, we have subtracted c-number terms and have set spectral parameters as $\xi \to 0$.

Now we consider the trigonometric CMW model as a degeneration of $\mathcal{H}_{\rm CMW}$ (1.1); we use the trigonometric solutions as in (2.11). After we redefine in the Dunkl operator $\hat{d}_j(\xi)$ (equation (3.3)) the coordinates $\exp(2\pi {\rm i} z_j) \to z_j$ and spectral parameters $\exp(2\pi {\rm i}\xi_j) \to \xi_j$, and set [17]

$$0 \ll \xi_1 \ll \xi_2 \ll \xi_3 \tag{3.9}$$

we obtain the *trigonometric* Dunkl operators as,

$$\hat{d}_j = z_j \frac{\partial}{\partial z_j} + a\left(\sum_{k<j} \theta_{kj} K_{jk} - \sum_{k>j} \theta_{jk} K_{jk}\right) - b\left(\sum_{\substack{k=1\\k\neq j}}^{3} \tilde{\theta}_{jk} L_{jk} - \sum_{\substack{k,l=1\\j\neq k\neq l\neq j}}^{3} \tilde{\theta}_{kl} L_{kl}\right) \tag{3.10}$$

where functions θ_{jk} and $\tilde{\theta}_{jk}$ are denoted as

$$\theta_{jk} = \frac{z_j}{z_j - z_k} \qquad \text{or} \qquad \frac{z_j^3}{z_j^3 - z_k^3} \tag{3.11}$$

$$\tilde{\theta}_{jk} = \begin{cases} \dfrac{z_l^2}{z_j z_k - z_l^2} & \text{for } j = 3 \text{ or } k = 3 \\ \dfrac{z_j z_k}{z_j z_k - z_l^2} & \text{otherwise.} \end{cases} \tag{3.12}$$

By construction, these trigonometric operators constitute an integrable family. Although Quesne also introduced the commuting trigonometric operators associated with the G_2 root system [10], our construction based on the cYBE (2.1) is suggestive, and is simply given as a degeneration of the *elliptic* case.

We find that the Dunkl operators satisfy the following commutation relations:

$$K_{12}\hat{d}_1 = \hat{d}_2 K_{12} - a \tag{3.13a}$$

$$K_{12}\hat{d}_2 = \hat{d}_1 K_{12} + a \tag{3.13b}$$

$$K_{12}\hat{d}_3 = \hat{d}_3 K_{12} \tag{3.13c}$$

$$L_{13}\hat{d}_1 = \tfrac{1}{3}(2\hat{d}_1 + 2\hat{d}_2 - \hat{d}_3)L_{13} + b \tag{3.13d}$$

$$L_{13}\hat{d}_2 = \tfrac{1}{3}(2\hat{d}_1 - \hat{d}_2 + 2\hat{d}_3)L_{13} - 2b \tag{3.13e}$$

$$L_{13}\hat{d}_3 = \tfrac{1}{3}(-\hat{d}_1 + 2\hat{d}_2 + 2\hat{d}_3)L_{13} + b. \tag{3.13f}$$

These are the defining relations of the degenerate affine Hecke algebra of type G_2 [19]. The conserved operators are calculated after tedious calculation as follows:

$$\mathcal{I}_1 = \sum_{j=1}^{3} z_j \frac{\partial}{\partial z_j}$$

$$\mathcal{I}_2 = \sum_{j=1}^{3}\left(z_j \frac{\partial}{\partial z_j}\right)^2 + \sum_{\substack{i,j=1\\i\neq j}}^{3} \frac{(z_i z_j)^\kappa}{(z_i^\kappa - z_j^\kappa)^2}(a\kappa P_{ij} - a^2) + 3\sum_{\substack{i,j,k=1\\i\neq j\neq k\neq i}}^{3} \frac{z_i z_j z_k^2}{(z_i z_j - z_k^2)^2}(b\tilde{P}_{ij} - b^2)$$

where $\kappa = 1$ or 3. When we introduce the coordinates as $z_j = \exp(2\pi {\rm i} x_j/L)$, the conserved operators $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively reduce to the total momentum and the Hamiltonian of the trigonometric CMW model [2],

$$\left(\frac{2\pi}{L}\right) \cdot \mathcal{I}_1 = -{\rm i}\sum_{i=1}^{3} \frac{\partial}{\partial x_j}$$

$$\left(\frac{2\pi}{L}\right)^2 \cdot \mathcal{I}_2 = -\sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} + \left(\frac{\pi}{L}\right)^2 \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{a^2 - a\kappa P_{ij}}{\sin^2 \frac{\pi}{L}\kappa(x_i - x_j)}$$
$$+3\left(\frac{\pi}{L}\right)^2 \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \frac{b^2 - b\tilde{P}_{ij}}{\sin^2 \frac{\pi}{L}(x_i + x_j - 2x_k)}.$$

3.2. NLS model of type-G_2

We shall treat the quantum integrability of the nonlinear Schrödinger model with two- and three-body interactions in the same manner. We use the Dunkl operator (3.3) with the singular solution (2.13), and for our later convenience we set the spectral parameters as in (3.9). In this region the Dunkl operator becomes independent of the spectral parameters, and is explicitly given by

$$\hat{d}_j = -\mathrm{i}\frac{\partial}{\partial z_j} + \mathrm{i}\frac{a}{2}\sum_{\substack{k=1\\ k\neq j}}^{3}(\varepsilon(z_{jk}) - \varepsilon_{jk})K_{jk}$$
$$+\mathrm{i}\frac{b}{2}\left(\sum_{\substack{k=1\\ k\neq j}}^{3}(\varepsilon(\tilde{z}_{jk}) - \tilde{\varepsilon}_{jk})L_{jk} - \sum_{\substack{k,l=1\\ j\neq k\neq l\neq j}}^{3}(\varepsilon(\tilde{z}_{kl}) - \tilde{\varepsilon}_{kl})L_{kl}\right). \qquad (3.14)$$

Here we have set $\alpha = \mathrm{i}a/2$ and $\beta = \mathrm{i}b/2$. Constants ε_{jk} and $\tilde{\varepsilon}_{jk}$ are defined as

$$\varepsilon_{jk} = \begin{cases} +1 & \text{for } j > k \\ -1 & \text{for } j < k \end{cases} \qquad (3.15)$$

$$\tilde{\varepsilon}_{jk} = \begin{cases} +1 & \text{for } j = 3 \text{ or } k = 3 \\ -1 & \text{otherwise.} \end{cases} \qquad (3.16)$$

One sees that operators $\hat{d}_j$ satisfy the same commutation relations (3.13) with the trigonometric Dunkl operators as follows:

$$K_{12}\hat{d}_1 = \hat{d}_2 K_{12} + \mathrm{i}a \qquad (3.17a)$$
$$K_{12}\hat{d}_2 = \hat{d}_1 K_{12} - \mathrm{i}a \qquad (3.17b)$$
$$K_{12}\hat{d}_3 = \hat{d}_3 K_{12} \qquad (3.17c)$$
$$L_{13}\hat{d}_1 = \tfrac{1}{3}(2\hat{d}_1 + 2\hat{d}_2 - \hat{d}_3)L_{13} - \mathrm{i}b \qquad (3.17d)$$
$$L_{13}\hat{d}_2 = \tfrac{1}{3}(2\hat{d}_1 - \hat{d}_2 + 2\hat{d}_3)L_{13} + 2\mathrm{i}b \qquad (3.17e)$$
$$L_{13}\hat{d}_3 = \tfrac{1}{3}(-\hat{d}_1 + 2\hat{d}_2 + 2\hat{d}_3)L_{13} - \mathrm{i}b. \qquad (3.17f)$$

For these Dunkl operators, we calculate the first two conserved operators $\mathcal{I}_n$ explicitly as follows:

$$\mathcal{I}_1 = -\mathrm{i}\sum_{j=1}^{3}\frac{\partial}{\partial z_j}$$
$$\mathcal{I}_2 = -\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2} + a\sum_{\substack{i,j=1\\ i\neq j}}^{3}\delta(z_{ij})P_{ij} + 3b\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3}\delta(\tilde{z}_{ij})\tilde{P}_{ij}.$$

We see that operator $\mathcal{I}_2$ coincides with the Hamiltonian of the three-body NLS model $\mathcal{H}NLS$ (1.2) and that operator $\mathcal{I}_1$ denotes a total momentum. This fact proves the quantum integrability of the coloured NLS model interacting via two- and three-body potentials.

4. Scattering matrix of the NLS model

In this section we shall study the scattering matrices for the NLS model $\mathcal{H}NLS$ associated with the root system of type G_2 (1.2). Since the conserved operators $\mathcal{I}_n$ (3.5) for the NLS model are given from the mutually commuting differential-difference operators $\hat{d}_j$ (3.14), the diagonalization of $\mathcal{I}_n$ is equivalent to that of the Dunkl operators $\hat{d}_j$. We shall diagonalize the Dunkl operator $\hat{d}_j$ (3.14) to calculate the scattering matrices.

As we have proved the quantum integrability of the NLS model, we suppose that the scattering matrix is factorizable. First we shall diagonalize the two-body Dunkl operators to determine the two-body scattering matrices, $S_{12}(k_1, k_2)$ [11]. Consider the following differential equations ($\boldsymbol{k} = (k_1, k_2)$):

$$\hat{d}_1 \Psi_{\boldsymbol{k}}(z_1, z_2) = k_1 \Psi_{\boldsymbol{k}}(z_1, z_2) \tag{4.1a}$$

$$\hat{d}_2 \Psi_{\boldsymbol{k}}(z_1, z_2) = k_2 \Psi_{\boldsymbol{k}}(z_1, z_2) \tag{4.1b}$$

where k_1 and k_2 are momenta, and the two-particle Dunkl operators $\hat{d}_j$ are given by

$$\hat{d}_1 = -\mathrm{i}\frac{\partial}{\partial z_1} + \mathrm{i}\frac{a}{2}(\varepsilon(z_{12}) + 1)\hat{s}_{12}$$

$$\hat{d}_2 = -\mathrm{i}\frac{\partial}{\partial z_2} + \mathrm{i}\frac{a}{2}(\varepsilon(z_{21}) - 1)\hat{s}_{12}.$$

The above eigenvalue problems (4.1) are solved exactly, and their solution is

$$\Psi_{\boldsymbol{k}}(z_1, z_2) = \frac{1 + \varepsilon(z_{12})}{2}\left(\mathrm{e}^{\mathrm{i}k_1 z_1 + \mathrm{i}k_2 z_2} + \frac{\mathrm{i}a}{k_1 - k_2 - \mathrm{i}a}\mathrm{e}^{\mathrm{i}k_2 z_1 + \mathrm{i}k_1 z_2}\right) + \frac{1 - \varepsilon(z_{12})}{2}\left(\frac{k_1 - k_2}{k_1 - k_2 - \mathrm{i}a}\mathrm{e}^{\mathrm{i}k_1 z_1 + \mathrm{i}k_2 z_2}\right). \tag{4.2}$$

This form of the wavefunction $\Psi_{\boldsymbol{k}}(z_1, z_2)$ indicates that the scattering matrix of the two particles is given by

$$S_{12}(k_1, k_2) = \frac{k_1 - k_2 + \mathrm{i}aP_{12}}{k_1 - k_2 - \mathrm{i}a}. \tag{4.3}$$

One sees that the two-body scattering matrix $S_{12}(k_1, k_2)$ satisfies the Yang–Baxter equation [18],

$$S_{12}(k_1, k_2)S_{13}(k_1, k_3)S_{23}(k_2, k_3) = S_{23}(k_2, k_3)S_{13}(k_1, k_3)S_{12}(k_1, k_2). \tag{4.4}$$

Next we shall calculate the three-body scattering matrix $\tilde{S}_{23}(k_1, k_2, k_3)$ due to the three-body interactions. Here, for brevity, we shall set $a = 0$ in the Dunkl operators for the NLS model (3.14), and consider the following differential equations ($\boldsymbol{k} = (k_1, k_2, k_3)$);

$$\hat{d}_1 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) = k_1 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) \tag{4.5a}$$

$$\hat{d}_2 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) = k_2 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) \tag{4.5b}$$

$$\hat{d}_3 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) = k_3 \Psi_{\boldsymbol{k}}(z_1, z_2, z_3) \tag{4.5c}$$

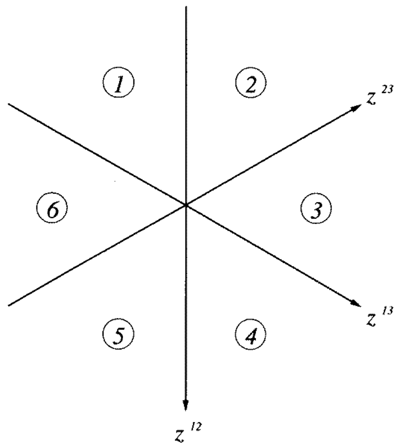

Figure 2. Divided regions for three-body scattering matrix.

where k_1, k_2, and k_3 are momenta, and the three-body Dunkl operators $\hat{d}_j$ are given by

$$\hat{d}_1 = -\mathrm{i}\frac{\partial}{\partial z_1} + \mathrm{i}\frac{b}{2}((\varepsilon(\tilde{z}_{13}) - 1)L_{13} + (\varepsilon(\tilde{z}_{12}) + 1)L_{12} - 2(\varepsilon(\tilde{z}_{23}) - 1)L_{23})$$
$$\hat{d}_2 = -\mathrm{i}\frac{\partial}{\partial z_2} + \mathrm{i}\frac{b}{2}((\varepsilon(\tilde{z}_{23}) - 1)L_{23} + (\varepsilon(\tilde{z}_{12}) + 1)L_{12} - 2(\varepsilon(\tilde{z}_{13}) - 1)L_{13})$$
$$\hat{d}_3 = -\mathrm{i}\frac{\partial}{\partial z_3} + \mathrm{i}\frac{b}{2}((\varepsilon(\tilde{z}_{23}) - 1)L_{23} + (\varepsilon(\tilde{z}_{13}) - 1)L_{13} - 2(\varepsilon(\tilde{z}_{12}) + 1)L_{12}).$$

To solve the differential equations, we assume that the eigenfunction $\Psi_{\boldsymbol{k}}(z_1, z_2, z_3)$ has a form

$$\Psi_{\boldsymbol{k}}(z_1, z_2, z_3) = \sum_{j=1}^{6} \chi_j(z_1, z_2, z_3)\psi_j(z_1, z_2, z_3) \tag{4.6}$$

where $\chi_j(z_1, z_2, z_3)$ denote the characteristic functions of six divided regions which follow from roots of the G_2 (figure 2); explicitly they are defined as

$$\chi_1(z_1, z_2, z_3) = \frac{1 - \varepsilon(\tilde{z}_{13})}{2}\frac{1 - \varepsilon(\tilde{z}_{12})}{2}$$
$$\chi_2(z_1, z_2, z_3) = \frac{1 + \varepsilon(\tilde{z}_{23})}{2}\frac{1 + \varepsilon(\tilde{z}_{12})}{2}$$
$$\chi_3(z_1, z_2, z_3) = \frac{1 - \varepsilon(\tilde{z}_{23})}{2}\frac{1 - \varepsilon(\tilde{z}_{13})}{2}$$
$$\chi_4(z_1, z_2, z_3) = \frac{1 + \varepsilon(\tilde{z}_{13})}{2}\frac{1 + \varepsilon(\tilde{z}_{12})}{2}$$
$$\chi_5(z_1, z_2, z_3) = \frac{1 - \varepsilon(\tilde{z}_{23})}{2}\frac{1 - \varepsilon(\tilde{z}_{12})}{2}$$
$$\chi_6(z_1, z_2, z_3) = \frac{1 + \varepsilon(\tilde{z}_{23})}{2}\frac{1 + \varepsilon(\tilde{z}_{13})}{2}.$$

Substituting the wave function (4.6) into the eigenvalue problems (4.5), we obtain differential equations for ψ_j. Among them, equations concerning ψ_5 and ψ_6 are written as follows:

- for $\tilde{z}_{23} > 0$ and $\tilde{z}_{13} > 0$,

$$-\mathrm{i}\frac{\partial}{\partial z_1}\psi_6(z_1, z_2, z_3) = k_1\psi_6(z_1, z_2, z_3) \tag{4.7a}$$

$$-\mathrm{i}\frac{\partial}{\partial z_2}\psi_6(z_1, z_2, z_3) = k_2\psi_6(z_1, z_2, z_3) \tag{4.7b}$$

$$-\mathrm{i}\frac{\partial}{\partial z_3}\psi_6(z_1, z_2, z_3) = k_3\psi_6(z_1, z_2, z_3) \tag{4.7c}$$

- for $\tilde{z}_{23} < 0$ and $\tilde{z}_{12} < 0$,

$$-\mathrm{i}\frac{\partial}{\partial z_1}\psi_5(z_1, z_2, z_3) + 2\mathrm{i}b\psi_6(2R - z_1, 2R - z_3, 2R - z_2) = k_1\psi_5(z_1, z_2, z_3) \tag{4.8a}$$

$$-\mathrm{i}\frac{\partial}{\partial z_2}\psi_5(z_1, z_2, z_3) - \mathrm{i}b\psi_6(2R - z_1, 2R - z_3, 2R - z_2) = k_2\psi_5(z_1, z_2, z_3) \tag{4.8b}$$

$$-\mathrm{i}\frac{\partial}{\partial z_3}\psi_5(z_1, z_2, z_3) - \mathrm{i}b\psi_6(2R - z_1, 2R - z_3, 2R - z_2) = k_3\psi_5(z_1, z_2, z_3) \tag{4.8c}$$

- for $\tilde{z}_{23} = 0$,

$$\psi_5(z_1, z_2, z_3) = \psi_6(z_1, z_2, z_3). \tag{4.9}$$

Under a condition for continuity (4.9), equations (4.7) and (4.8) are solved as

$$\psi_6(z_1, z_2, z_3) = \frac{\tilde{k}_{23}}{\tilde{k}_{23} + 3\mathrm{i}b}\mathrm{e}^{\mathrm{i}k_1z_1+\mathrm{i}k_2z_2+\mathrm{i}k_3z_3} \tag{4.10a}$$

$$\psi_5(z_1, z_2, z_3) = \mathrm{e}^{\mathrm{i}k_1z_1+\mathrm{i}k_2z_2+\mathrm{i}k_3z_3} - \frac{3\mathrm{i}b}{\tilde{k}_{23} + 3\mathrm{i}b}L_{23}\mathrm{e}^{\mathrm{i}k_1z_1+\mathrm{i}k_2z_2+\mathrm{i}k_3z_3}. \tag{4.10b}$$

Comparing the coefficients of this wavefunction, we conclude that the scattering matrix for the three-body interaction are given as

$$\tilde{S}_{23}(k_1, k_2, k_3) = \frac{\tilde{k}_{23} - 3\mathrm{i}b\tilde{P}_{23}}{\tilde{k}_{23} + 3\mathrm{i}b}. \tag{4.11}$$

We note that the three-body scattering matrix $\tilde{S}_{ij}(k_1, k_2, k_3)$ also satisfies the Yang–Baxter equation,

$$\begin{aligned}&\tilde{S}_{12}(k_1, k_2, k_3)\tilde{S}_{13}(-k_1, -k_2, -k_3)\tilde{S}_{23}(k_1, k_2, k_3)\\&\quad = \tilde{S}_{23}(k_1, k_2, k_3)\tilde{S}_{13}(-k_1, -k_2, -k_3)\tilde{S}_{12}(k_1, k_2, k_3).\end{aligned} \tag{4.12}$$

Moreover, the scattering matrices $S_{ij}(k_1, k_2)$ (4.3) and $\tilde{S}_{ij}(k_1, k_2, k_3)$ (4.11) satisfy the Yang–Baxter equation for the root system of type G_2 [19],

$$\begin{aligned}&S_{23}(k_2, k_3)\tilde{S}_{12}(k_1, k_2, k_3)S_{13}(k_1, k_3)\tilde{S}_{23}(-k_1, -k_2, -k_3)S_{12}(k_1, k_2)\tilde{S}_{13}(k_1, k_2, k_3)\\&\quad = \tilde{S}_{13}(k_1, k_2, k_3)S_{12}(k_1, k_2)\tilde{S}_{23}(-k_1, -k_2, -k_3)\\&\qquad \times S_{13}(k_1, k_3)\tilde{S}_{12}(k_1, k_2, k_3)S_{23}(k_2, k_3).\end{aligned} \tag{4.13}$$

5. Concluding remarks

In this paper we have studied the integrable three-body problem with two- and three-body interactions, which is mathematically related with the root systems of type G_2. The crucial tools are the classical Yang–Baxter equation (2.1) for the root system G_2, whose solutions we have regarded as operators acting on the functional space. Following the idea of [11], we have proposed a systematic way to construct a set of the Dunkl operators, i.e. a set of the integrable differential-difference operators, associated with the root system of type G_2 using the operator-valued solutions of the cYBE. Generally the Dunkl operator can be written as ($j = 1, 2, 3$)

$$\hat{d}_j(\xi) = -\mathrm{i}\frac{\partial}{\partial z_j} + \sum_{\substack{k=1\\k\neq j}}^{3} F(\xi_{jk}, z_{jk})K_{jk} + \sum_{\substack{k=1\\k\neq j}}^{3} G(\tilde{\xi}_{jk}, \tilde{z}_{jk})L_{jk} - \sum_{\substack{k,l=1\\j\neq k\neq l\neq j}}^{3} G(\tilde{\xi}_{kl}, \tilde{z}_{kl})L_{kl} \qquad (5.1)$$

where functions $F(\xi, z)$ and $G(\xi, z)$ are solutions of the functional equations (2.9). Parameters $\{\xi_j\}$ play a role of the spectral parameters and take arbitrary values.

As examples of the above Dunkl operators, we have introduced two sets of operators; elliptic and singular operators. We have shown that these Dunkl operators respectively prove the integrability of the CMW model and the NLS model. It has been pointed out that the Dunkl operators for both the trigonometric CMW model and the NLS model constitute the same algebra, called the degenerate affine Hecke algebra of type G_2. We have also shown that the scattering matrices of the NLS model, which are given by diagonalizing the Dunkl operators, satisfy the Yang–Baxter equation associated with the root system of type G_2. The diagonalization of the trigonometric Dunkl operators (3.10) could be done in the same way, and the eigenpolynomial should be related with the *non-symmetric* Jack polynomials.

We hope in a forthcoming paper to study the lattice analogue of both the CMW model and the NLS model by use of operator-valued solutions of the YBE associated with the root system of type G_2.

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